Qualitative Calculus and Qualitative Physics: Theory & Application

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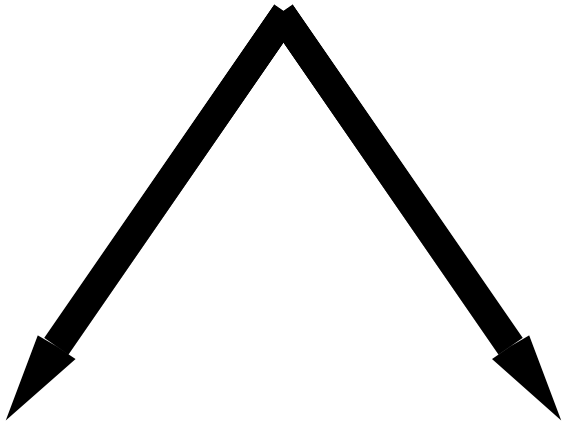
Qualitative Physics



Naive Physics	**Engineer's Physics**
Commonsense Physics	
Mac Carthy	**De Kleer**
Pat Hayes	**Williams**
Forbus	**Kuipers**
	Weld

References

- **Mac Carthy & Pat Hayes, 1969**
- **Pat Hayes** ***The Naive Physics Manifesto*** **1977**
- **Pat Hayes** ***The Second Naive Physics Manifesto*** **1983**

Artificial Intelligence Vol. 24, December 1984:

- **De Kleer & Brown** ***A Qualitative Physics Based on Confluences***
- **De Kleer** ***How Circuits Work***
- **Forbus** ***Qualitative Process Theory***
- **Kuipers** ***Qualitative Simulation***
- **Williams** ***A Qualitative Analysis of MOS Circuits***
- **Iwasaki & Simon** ***Theories of Causal Ordering*** **1986**

Qualitative Physics Workshop
Urbana-Champaign (Illinois), May 1987

Second Qualitative Physics Workshop

Paris, July 1988

AAAI 88: ˜ 25 papers

MODELS

Signs

- **Introduced by the Economists (Samuelson 1947,Lancaster 1962)**

- **Used in Qualitative Physics under various formalisms**

- **Used in Control Theory (Travé 1986-1988)**

Orders of magnitude

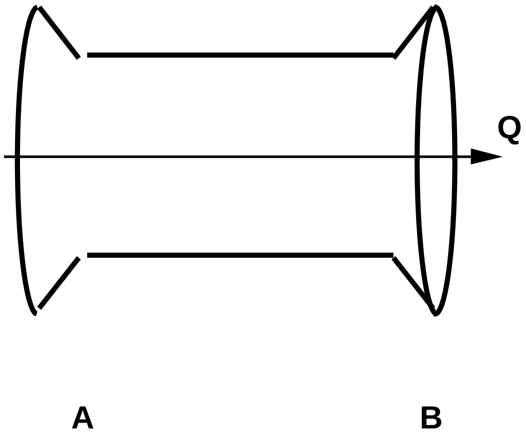
- **Raiman AAAI 1986**

- **Dan Weld's Exaggeration**

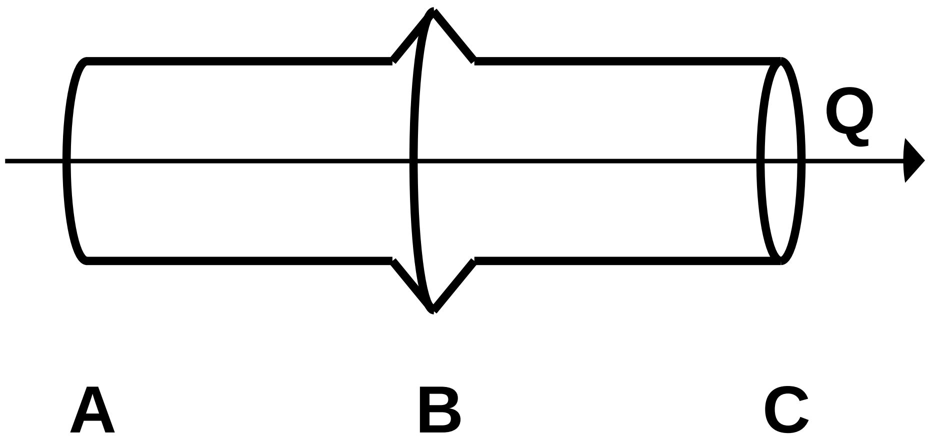
PROBLEMS

- **"Comparative statics" = Linear equations**

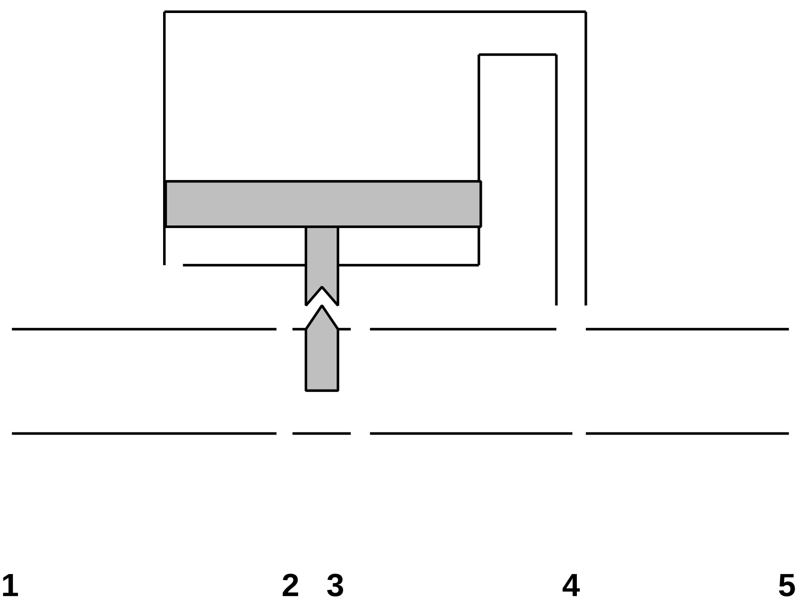
- **Qualitative dynamics = Differential equations**



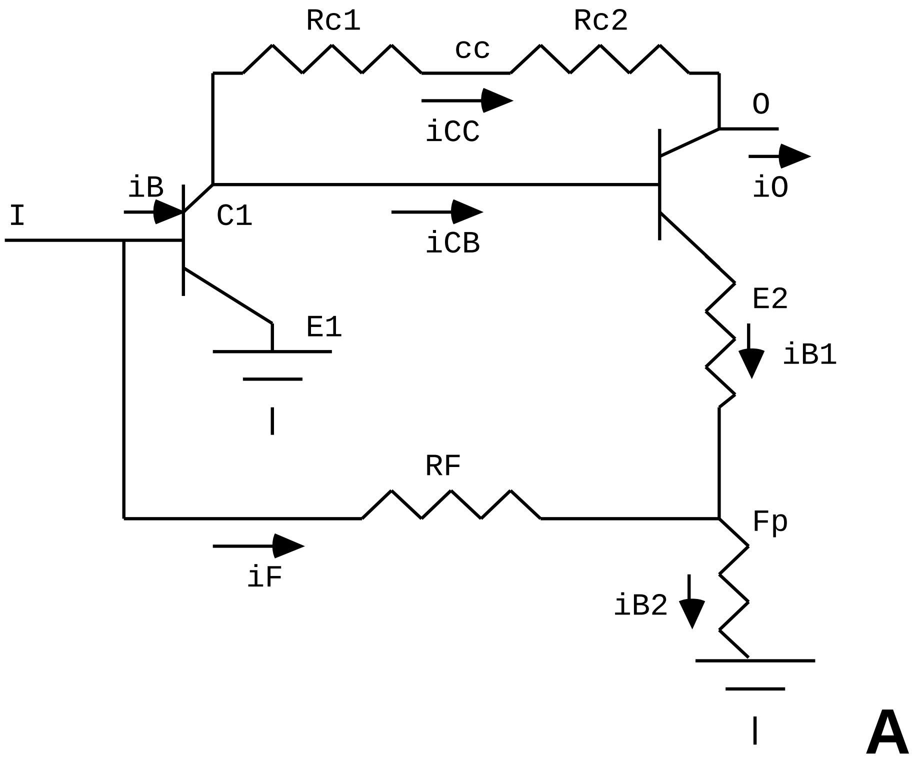
A pipe



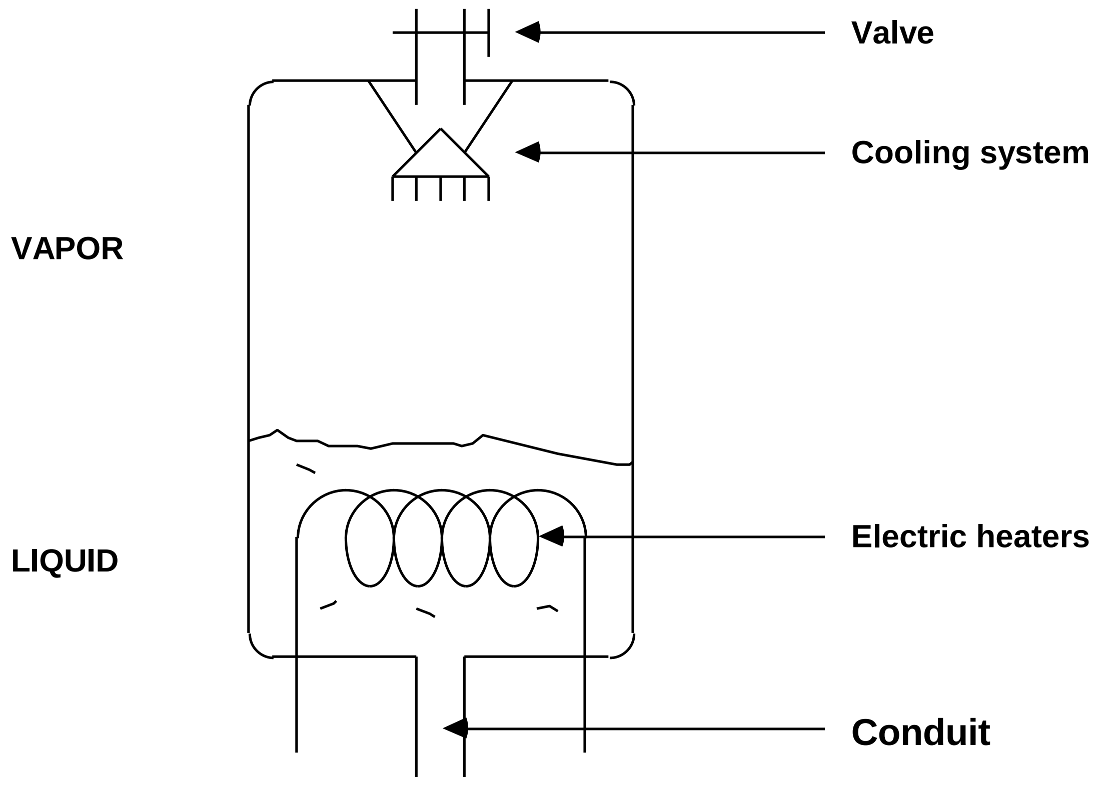
Two connected pipes



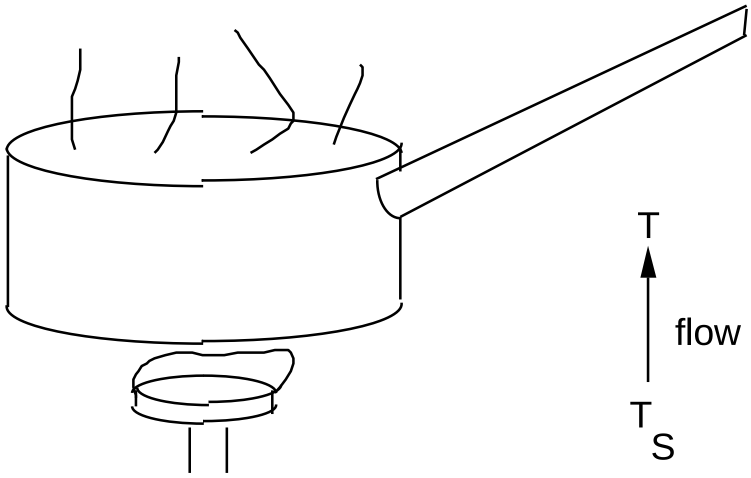
The pressure regulator



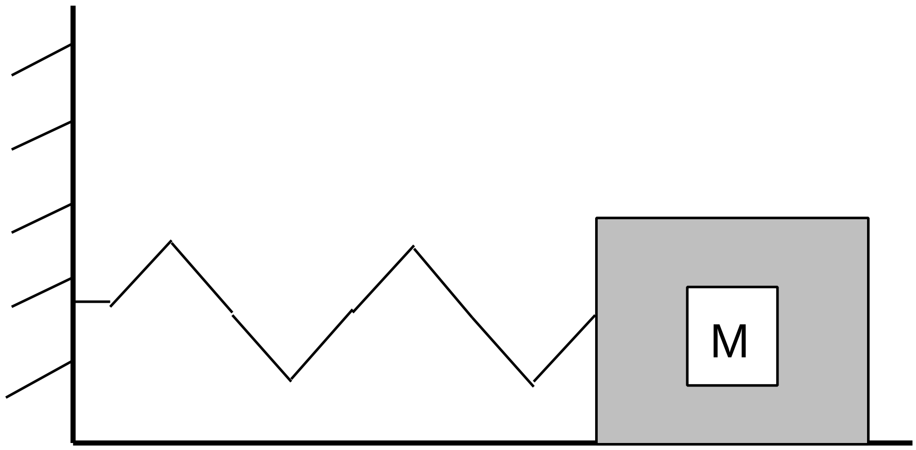
CE-feedback



The pressurizer of a nuclear power plant



Heading a saucepan



A simple oscillator

PV = nRT

®

[dP] + [dV] - [dn] - [dT] ˜ 0

or

?P + ?V - ?n - ?T ˜ 0

Confluence* or *Qualitative Equation

?X = [dX] = sign of dX or DX, i.e.

- **Sign of the variation dX of X during an infinitely small time interval.**

- **Sign of the variation DX of X between two distinct states.**

?X is called a *qualitative derivative*

<u>Remark</u>: In general, we work under the *quasi-static assumption*, i.e.
***In any transformation, a system passes through an infinite number of infinitely close equilibrium states*.**

$$[dP] + [dV] - [dn] - [dT] \sim 0$$

[dP]=+, [dV]=0, [dn]=-, [dT]=- does not satisfy the confluence

[dP]=+, [dV]=0, [dn]=- ==> [dT]=+

[dP]=+, [dV]=+ ==> the confluence is satisfied

A confluence is a necessary condition which must be satisfied by the signs of the physical quantities it involves

A Qualitative Calculus
Based on Signs

S = { + , 0 , - , ? }

Addition and multiplication:

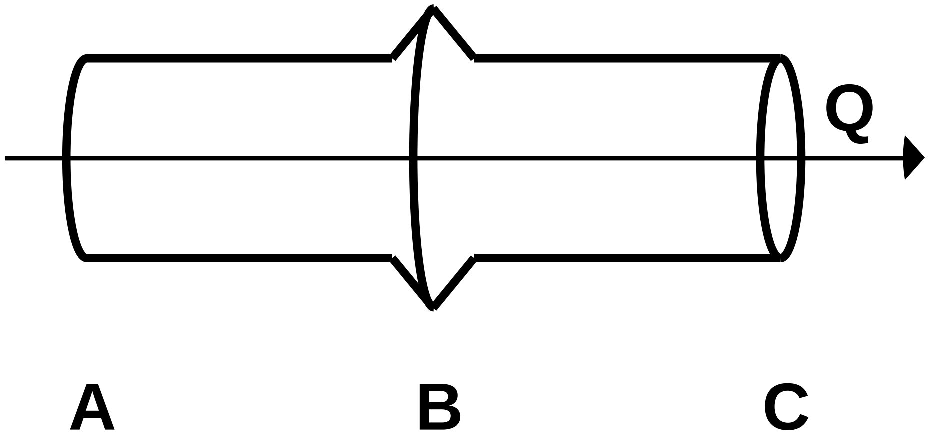
+	0	+	-	?
0	0	+	-	?
+	+	+	?	?
-	-	?	-	?
?	?	?	?	?

*	0	+	-	?
0	0	0	0	0
+	0	+	-	?
-	0	-	+	?
?	0	?	?	?

Qualitative equality:

For all a, b belonging to S:

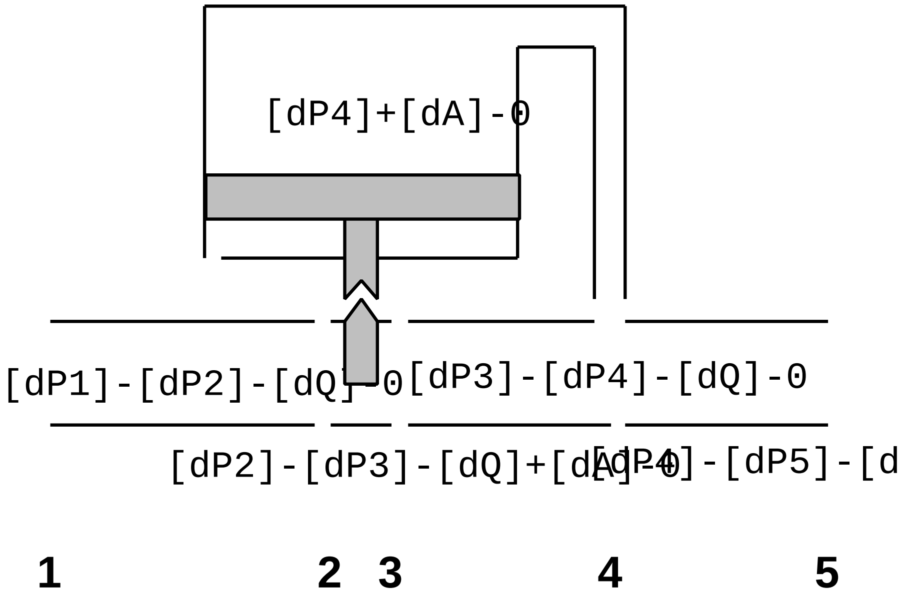
a ˜ b   iff   a = b
or
a = ?
or
b = ?



Two connected pipes

$$?P_A - ?P_B - ?Q \;\tilde{}\; 0 \qquad (1)$$

$$?P_B - ?P_C - ?Q \;\tilde{}\; 0 \quad (2)$$



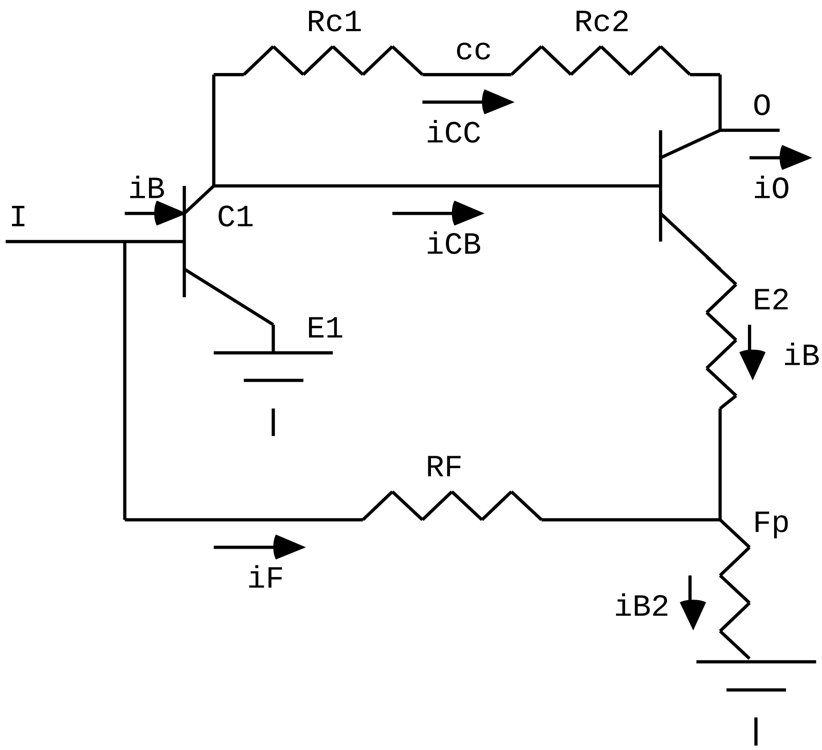
$?P_1 - ?P_2 - ?Q \sim 0$ (1)

$?P_2 - ?P_3 - ?Q + ?A \sim 0$ (2)

$?P_3 - ?P_4 - ?Q \sim 0$ (3)

$?P_4 - ?P_5 - ?Q \sim 0$ (4)

$?P_4 + ?A \sim 0$ (5)



Transistor Q_1:

$?v_I-?i_B$ ˜ 0 $?v_I-?i_{C1}$ ˜ 0

$?v_I-?i_{E1}$ ˜ 0

Transistor Q_2:

$?v_{C1E2}-?i_{CB}$ ˜ 0 $?v_{C1E2}-?i_{B1}$ ˜ 0

Ohm's law:

$?v_I-?v_{FP}-?i_F$ ˜ 0 Ohm(I,FP)

$?v_{E2}-?v_{FP}-?v_{C1E2}$ ˜ 0 Ohm(E2,FP)

$?v_{FP}-?i_{B2}$ ˜ 0 Ohm(FP,Ground)

$?v_{C1}-?i_{CC}$ ˜ 0 Ohm(C1,CC)

KCL:

$?i_I-?i_B-?i_F$ ˜ 0 KCL(I)

$?i_{C1}-?i_{CC1}-?i_{CB}$ ˜ 0 KCL(C1)

$?i_{B2}-?i_F-?i_{B1}$ ˜ 0 KCL(FP)

Drop of potential:

$?v_{C1E2}-?v_{C1}+?v_{E2}$ ˜ 0 PD(C1,E2)

Con	$= B_{11} + B_{12}*(W_1+W_2) + B_{13}*Profit + B_{14}*Profit(-1)$
W_1	$= K*W_1(-1) + B_{21} + B_{22}(Income+Tax-W_2) + B_{23}*(Income+Tax-W_2)(-1) + B_{24}*Time$
Income	= Con + Invest + Gov - Tax
Profit	$= Income - W_1 - W_2$

when

Con	= Domestic consumption
Gov	= Public expenditures
Income	= Gross domestic product
Invest	= Investments
Profit	= Profit
Tax	= Tax
W_1	= Private sector wages
W_2	= Public sector wages

$B_{ij}, K > 0$

X(-1) = X(Time-1)

What is the effect of an increase or a decrease in the "governmental variables" (Gov,Invest,Tax,W_2)?

1) Replace Income by Con+Invest+Gov-Tax and Profit by Con+Invest+Gov-Tax-W_1-W_2

2) Consider ?Con and ?W_1 caused by {?Gov,?Invest,?Tax,?W_2}.

$$-A.?W_1 + C_{11}.?Con = A.?W_2 + B_{13}.?Gov - B_{13}.?Tax + B_{13}.?Invest$$

$$?W_1 - B_{22}.?Con = -B_{22}.?W_2 + B_{22}.?Gov + B_{22}.?Invest$$

(Here $C_{11}=1-B_{13}$ and $A=B_{12}-B_{13}$)

If one denotes ?X=sign(?X), a=sign(A), and if one assumes $C_{11}>0$:

$$-a.?W_1 + ?Con \,\tilde{}\; a.?W_2 + ?Gov - ?Tax + ?Invest \quad (1)$$

$$?W_1 - ?Con \,\tilde{}\; -?W_2 + ?Gov + ?Invest \quad (2)$$

For example, if a=-
and if ?Tax=+ and ?Gov=?W_2=?Invest=0
then ?W_1=- and ?Con=-

Qualitative Linear Systems

• QLS = A qualitative linear system *not involving a quantity and one of its derivatives at the same time* (otherwise, one gets a *Qualitative Linear Differential System*).

• Solving a QLS

AX ˜ B

consists of finding vectors X without any ? component

• Let X_0 be a solution of a QLS AX ˜ B. Then, for any real vector X'_0 with the sign pattern of X_0, there is a matrix A' and a vector B' with the sign patterns of A and B such that $A'X'_0 = B'$.

• In practical terms, QLSs stem from:

? A set of real equations (possibly non-linear)

? A real differential system (comparative statics).

? A set of graphical constraints

Hard components

- **For a real linear system:**
  - **? There is no solution**
  - **? There is a single solution**
  - **? There is an infinite number of solutions.**

--> *The unicity problem is stated in terms of a global solution vector.*

- **For example, assume that a=- and ?Gov=+:**

$?W_1 + ?Con$	˜	+	(1)
$?W_1 - ?Con$	˜	+	(2)

Then $?W_1$=+, but ?Con remains ambiguous.

- **In a QLS, a component:**
  1) **is a hard component**
  2) **has solutions + and -, but not 0**
  3) **has solutions +, 0 and -.**

***Example of case 2: if* a=+ and if all the input variables remain steady, one gets:**

$-?W_1 + ?Con$	˜	0	(1)
$?W_1 - ?Con$	˜	0	(2)

and the solution set is

$?W_1$=?Con=±

Qualitative Rank

• **<u>Independant qualitative vectors</u>:** ***Let $V_1,...,V_n$ be some qualitative vectors of the same size. We say that they are independant iff for any $a_1,...,a_n$ all different from ?, the relation $a_1V_1+...+a_nV_n$ ˜ 0 implies $a_1=...=a_n=0$.***

• **<u>Qualitative rank</u>:**

? The <u>rank</u> of a qualitative matrix A is the maximum number of independant column vectors.

? A matrix A has full rank iff the QLS AX ˜ 0 has the single solution X=0.

? A QLS AX ˜ B is stationary iff matrix A has full rank.

• **<u>Qualitative rank and hard components</u>:** ***Let AX ˜ B be a QLS with a hard component x_j. Then there is a subsystem with full rank involving x_j.***

Qualitative determinant

• **<u>Full rank and determinant</u>: A square matrix A is not a full rank matrix iff Det(A)˜ 0.**

• **<u>Qualitative Cramer's Formula</u>: Let AX ˜ B be a non decomposable square QLS such that Det(A)?0. Let $A_{j/B}$ be the matrix deduced from A by substituting vector B for its j^{th} column. Then, for any a_j Î {+,0,-} such that**

$$a_j \text{˜} Det(A).Det(A_{j/B}),$$

there exists a solution vector X such that its j^{th} component $x_j = a_j$.

(A square matrix A is non decomposable if it cannot be matched by permuting its rows and columns to the form:

$$\begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix}$$

when A_1 and A_2 are square matrices)

Soft Components

• Several algorithms have been proposed to solve this problem

   ? De Kleer & Brown, 1984

   ? Travé & Kaskurewicz, 1986

• But the structure of the solution set has not been yet investigated.

The Qualitative Resolution Rule

Let x, y, z, a, b be in S = {+ , 0 , - , ?} such that

$x + y \sim a \qquad (1)$

$-x + z \sim b \qquad (2)$

Then, <u>if x is different from ?</u>

$y + z \sim a + b \qquad (3)$

In a practical way:

- **x stands for a quantity**
- **(1) and (2) are two confluences**
- **y and z are two expressions not involving the same variable with opposite coefficients**

"One can eliminate a variable by adding or subtracting two confluences, provided that no other variable is eliminated at the same time."

Example:

Adding the two equations:

$?W_1$ + ?Con ˜ -$?W_2$ + ?Gov - ?Tax + ?Invest (1)
$?W_1$ - ?Con ˜ -$?W_2$ + ?Gov + ?Invest (2)

$?W_1$ ˜ -$?W_2$ + ?Gov - ?Tax + ?Invest (3)

Subtracting the two equations:

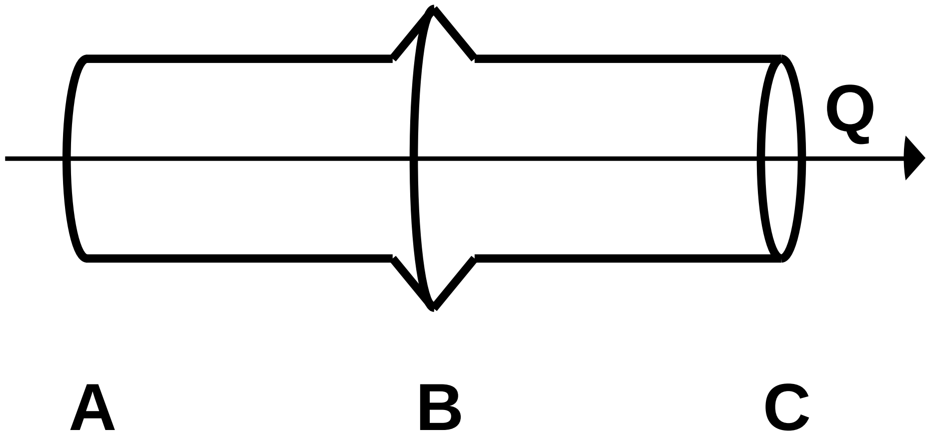
$?W_1$ + ?Con ˜ -$?W_2$ + ?Gov - ?Tax + ?Invest (1)
$?W_1$ - ?Con ˜ -$?W_2$ + ?Gov + ?Invest (2)

?Con ˜ ??W_2 +??Gov - ?Tax +??Invest (4)

$?W_1$ is a hard component iff
-$?W_2$ + ?Gov - ?Tax + ?Invest ? ?

?Con is a hard component iff
$?W_2$ = ?Gov = ?Invest = 0

Is the Sum of Two Pipes a Pipe?

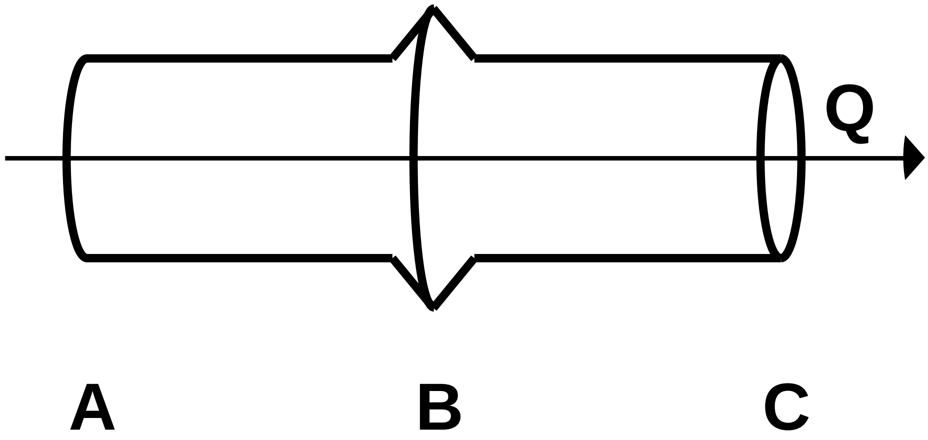


Two connected pipes

$$?P_{A} - ?P_{B} - ?Q \;\tilde{}\; 0 \quad (1)$$

$$?P_{B} - ?P_{C} - ?Q \;\tilde{}\; 0 \quad (2)$$

$$?P_{A} - ?P_{C} - ?Q \;\tilde{}\; 0 \quad (3)$$



y

x

a

ŽPA - ŽPB - ŽQ - 0 (1)

\+ ŽPB - ŽPC - ŽQ - 0 (2)

-x

b

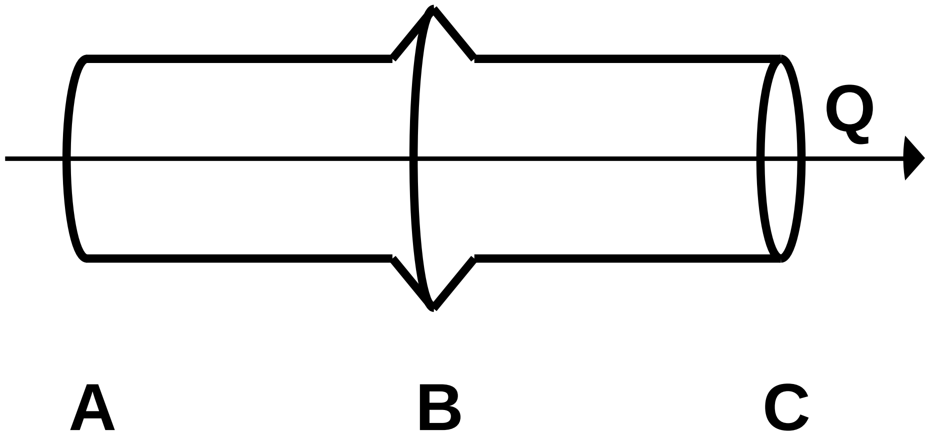
z

® ŽPA - ŽQ - ŽPC - ŽQ - 0

t + t = t

\- ŽQ

Þ ŽPA - ŽPC - ŽQ - 0 (3)=(1)+(2)



y x a

ŽPA - ŽPB - ŽQ - 0 (1)

\- ŽPB - ŽPC - ŽQ - 0 (2)

-z x -b

® ŽPA - ŽPB - ŽPB + ŽPC - 0

t + t = t

\- ŽPB

Þ ŽPA - ŽPB + ŽPC - 0 (4)=(1)-(2)

Consequences for a Simulation Task

<u>**Suppose that**</u> **?P$_A$=+**

?P$_C$=+

<u>Global relations + propagation rules:</u>

ŽPB - ŽPA + ŽPC → ŽPB = +

+ + +

ŽQ - ŽPA - ŽPC → ŽQ remains

+ - + ambiguous

<u>Initial model + propagation rules:</u>

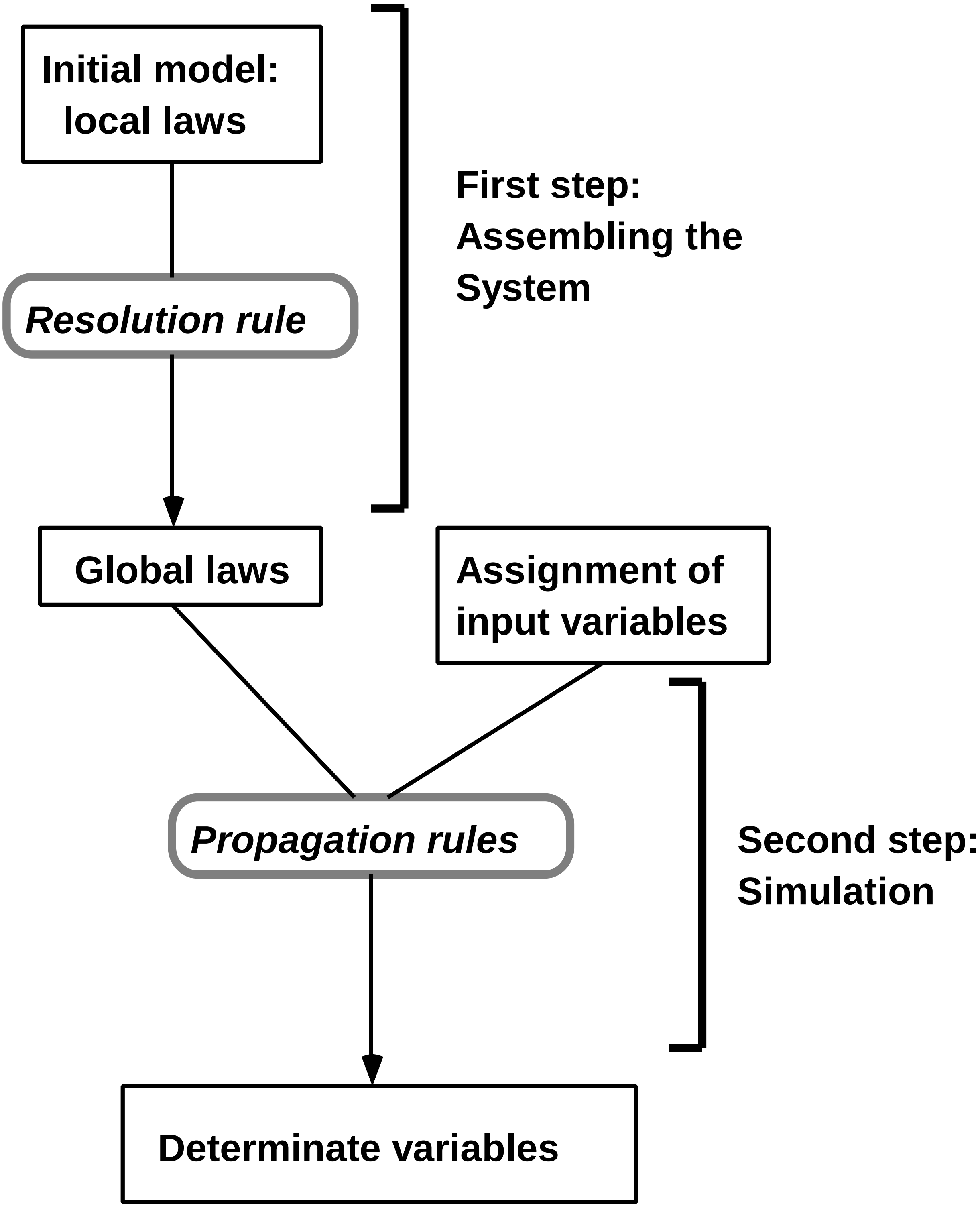
ŽPA - ŽPB - ŽQ - 0 → - ŽPB - ŽQ - -

+

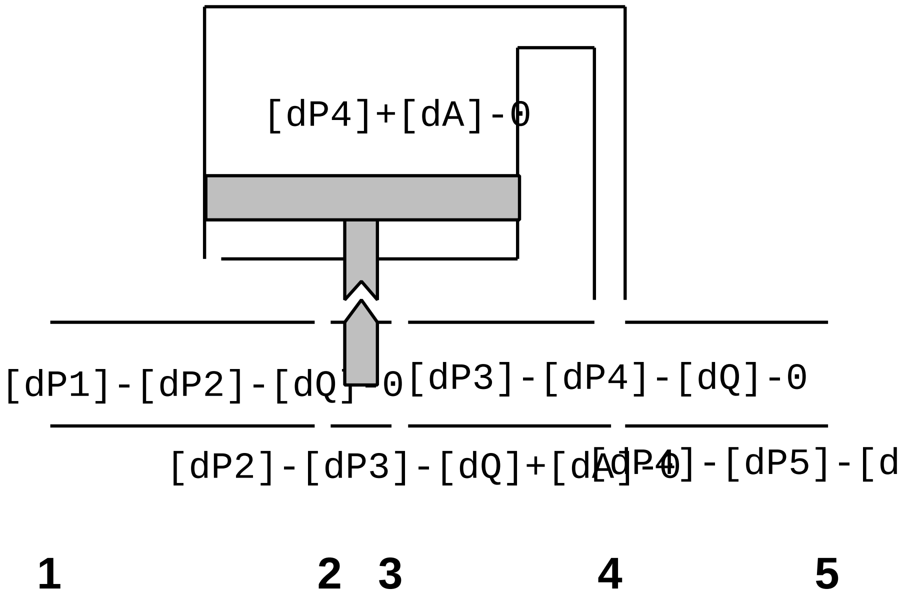
ŽPB - ŽPC - ŽQ - 0 → ŽPB - ŽQ - +

-

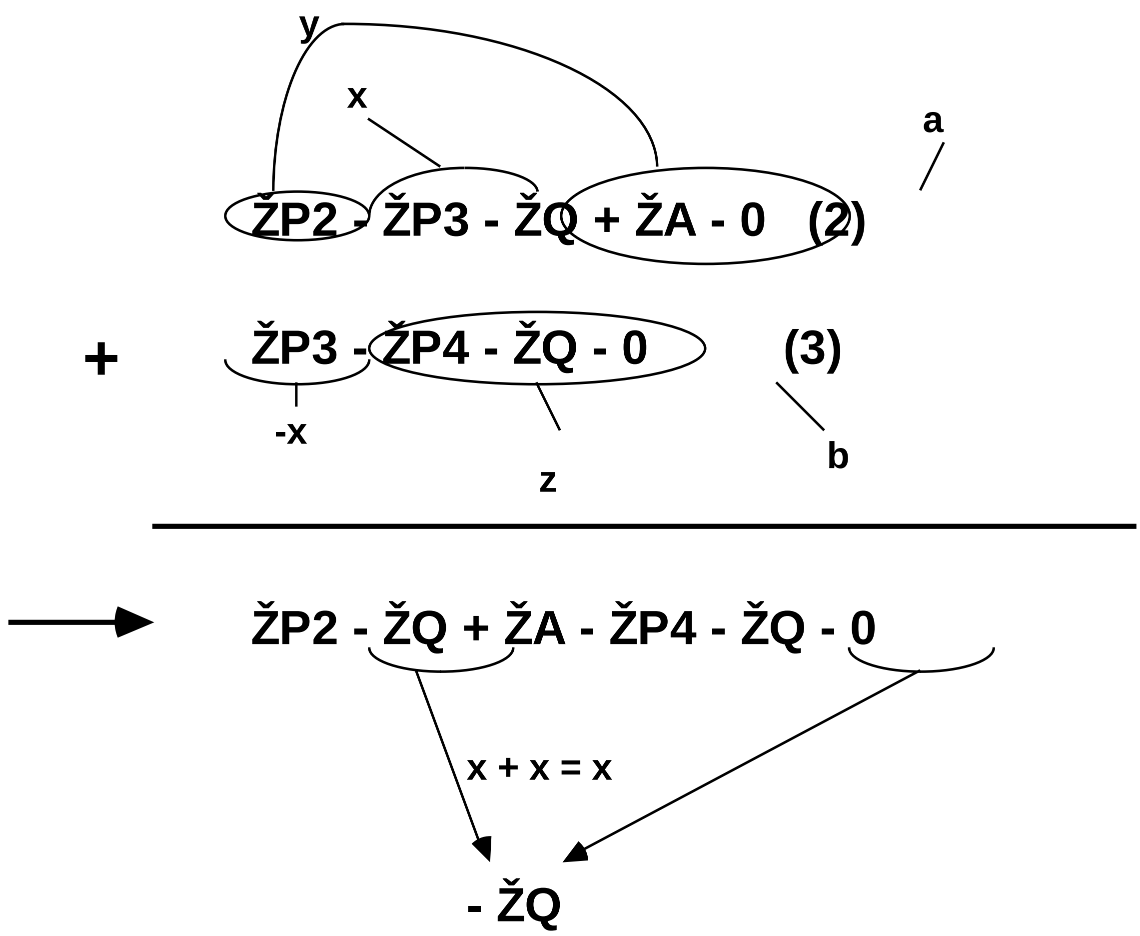
Assembling a System



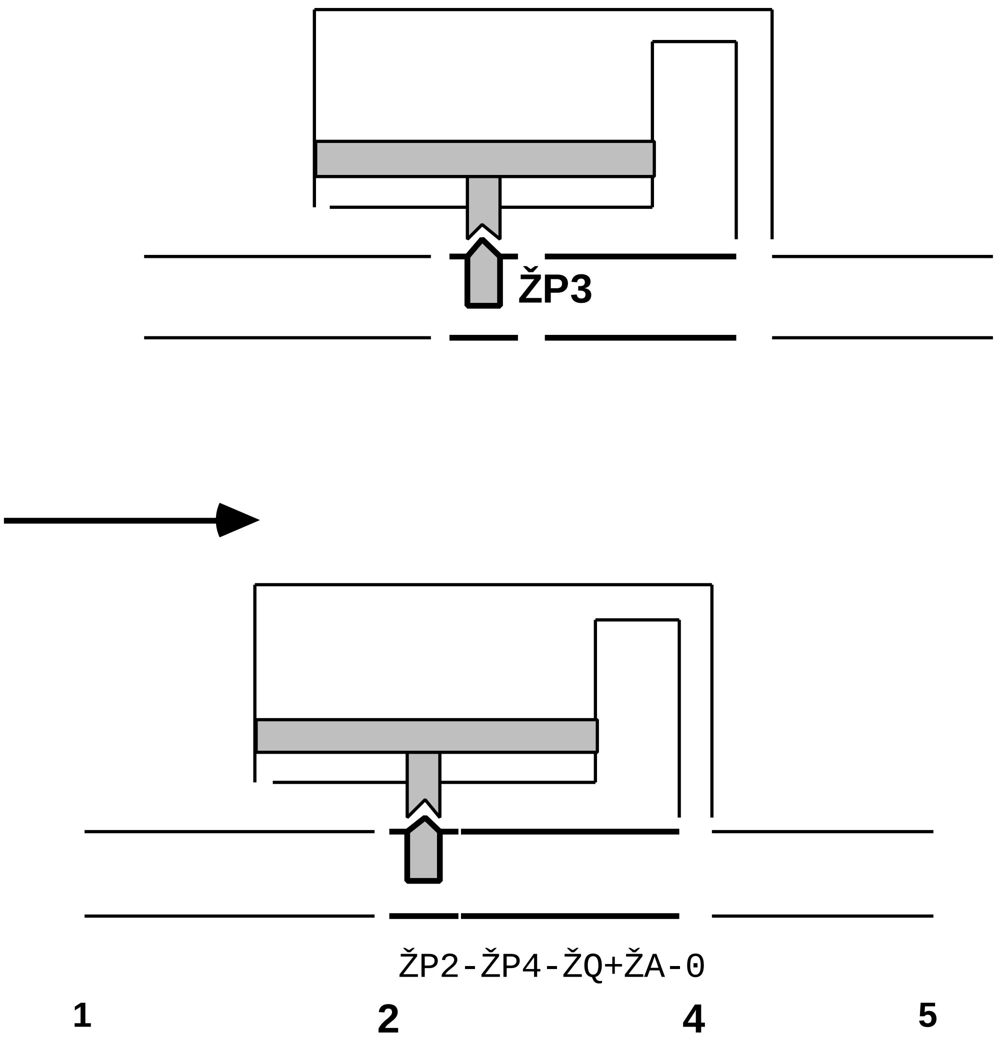
Second example:



$?P_1$ $-?P_2$ $-?Q$ ˜ 0 (1)
$?P_2$ $-?P_3$ $-?Q$ $+?A$ ˜ 0 (2)
$?P_3$ $-?P_4$ $-?Q$ ˜ 0 (3)
$?P_4$ $-?P_5$ $-?Q$ ˜ 0 (4)
$?P_4$ $+?A$ ˜ 0 (5)


y
x
a
ŽP2 - ŽP3 - ŽQ + ŽA - 0 (2)
+ ŽP3 - ŽP4 - ŽQ - 0 (3)
-x
z
b
ŽP2 - ŽQ + ŽA - ŽP4 - ŽQ - 0
x + x = x
- ŽQ

The resolution rule is based on a physical interpretation: it combines local behavioral descriptions into more global ones.



Recursively applying the resolution rule eventually provides direct relations - called <u>assemblages</u> - linking the internal variables and some selected reference variables (e.g., the input variables)

Reference variables = input variables

$?P_2 \sim ?P_1 + ?P_5$ (SA1)
$?P_4 \sim ?P_1 + ?P_5$ (SA2)
$?A \sim -?P_1 - ?P_5$ (SA3)
$?Q \sim ?P_1 - ?P_5$ (SA4)
$?P_3 \sim ?P_1 + ?\ ?P_5$ (SA5)

Reference variables = {?A,?Q}

$?P_1 \sim -?A + ?Q$
$?P_2 \sim -?A + ?Q$
$?P_3 \sim -?A + ?Q$
$?P_4 \sim -?A$
$?P_5 \sim -?A - ?Q$

Obtaining a task-oriented assemblage makes the corresponding simulation task straightforward.

Scanning the resolution rule

Proof

Quasi-transitivity of qualitative equality:

If

a ˜ b and b ˜ c

and b ? ?

then

b ˜ c

Compatibility of addition and qualitative equality:

a + b ˜ c is equivalent to a ˜ c - b

Proof:

x + y ˜ a ® y - a ˜ x

-x + z ˜ b ® x ˜ z - b

y - a ˜ z - b

®

y + z ˜ a + b

Let $x + E_1$ ˜ a and $-x + E_2$ ˜ b be two confluences such that x is a variable and E_1 and E_2 are two linear expressions not involving the same variable with opposite coefficients.
Then E_3 ˜ $a + b$ is a valid confluence, when E_3 is the same expression as E_1+E_2 but with no repeated variable.

$?P_2$ - $?P_3$ - ?Q + ?A ˜ 0 (2)
+ $?P_4$ + ?A ˜ 0 (5)

® $?P_2$ - $?P_3$ - ?Q + $?P_4$ ˜ 0 (6)=(2)+(5)

$?P_2$ - $?P_3$ + $?P_4$ - ?Q ˜ 0 (6)
+ $?P_3$ - $?P_4$ - ?Q ˜ 0 (3)

® $?P_2$ - $?P_4$ - ?Q ˜ 0 (7)=(6)+(3)

Performing (6) - (3) is impossible
(this would eliminate two variables
at the same time)

Completeness properties of the qualitative resolution rule

Definition of an assemblage:

Let C be a set of confluences, w_i be selected reference variables and v_j the remaining ones. A set of global laws A is called an assemblage for the reference variables w_i iff for each assignment of the reference variables $w_i=a_i$, as soon as the model imposes the value b_j to the internal variable v_j, then the basic propagation rules can deduce $v_j \tilde{} \, b_j$ from the assemblage.

(Consequently, if $v_j \tilde{}$?, then v_j is not determinate)

<u>Completeness</u> = Obtaining an assemblage

Qualitative resolution is complete (at least, in the square case):

If A X ˜ B is a square qualitative linear system (QLS) and if the j^{th} component x_j of X is determinate (and has the value a_j), then the qualitative resolution rule finds out in a finite number of steps the equation x_j ˜ a_j

® **The resolution rule always finds out an assemblage**

The proof requires notions such as:

- **qualitative determinant**
- **qualitative rank**
- **maximal matrices with full rank ...**

The general resolution rule is needed for completeness

<u>**Ritschard's rule (1983)**</u>**:**

Let $x + E_1 \sim a$ and $-x + E_2 \sim b$ be two confluences such that x is a variable and E_1 and E_2 are two linear expressions not involving the same variable with opposite coefficients. <u>Assume that all the variables involved in E_2 are also involved in E_1.</u>
Then $E_3 \sim a + b$ is a valid confluence, when E_3 is the same expression as E_1+E_2 but with no repeated variable. <u>Moreover, if a + b = b, then substituting confluence (C_3) for confluence (C_1) provides an equivalent set of confluences.</u>

Ritschard claimed a completeness result concerning this rule.

Counter-example:

$$
\begin{aligned}
y + z + t &\sim 0\\
x - z + t &\sim 0\\
x + y - t &\sim 0\\
x - y + z &\sim 0
\end{aligned}
$$

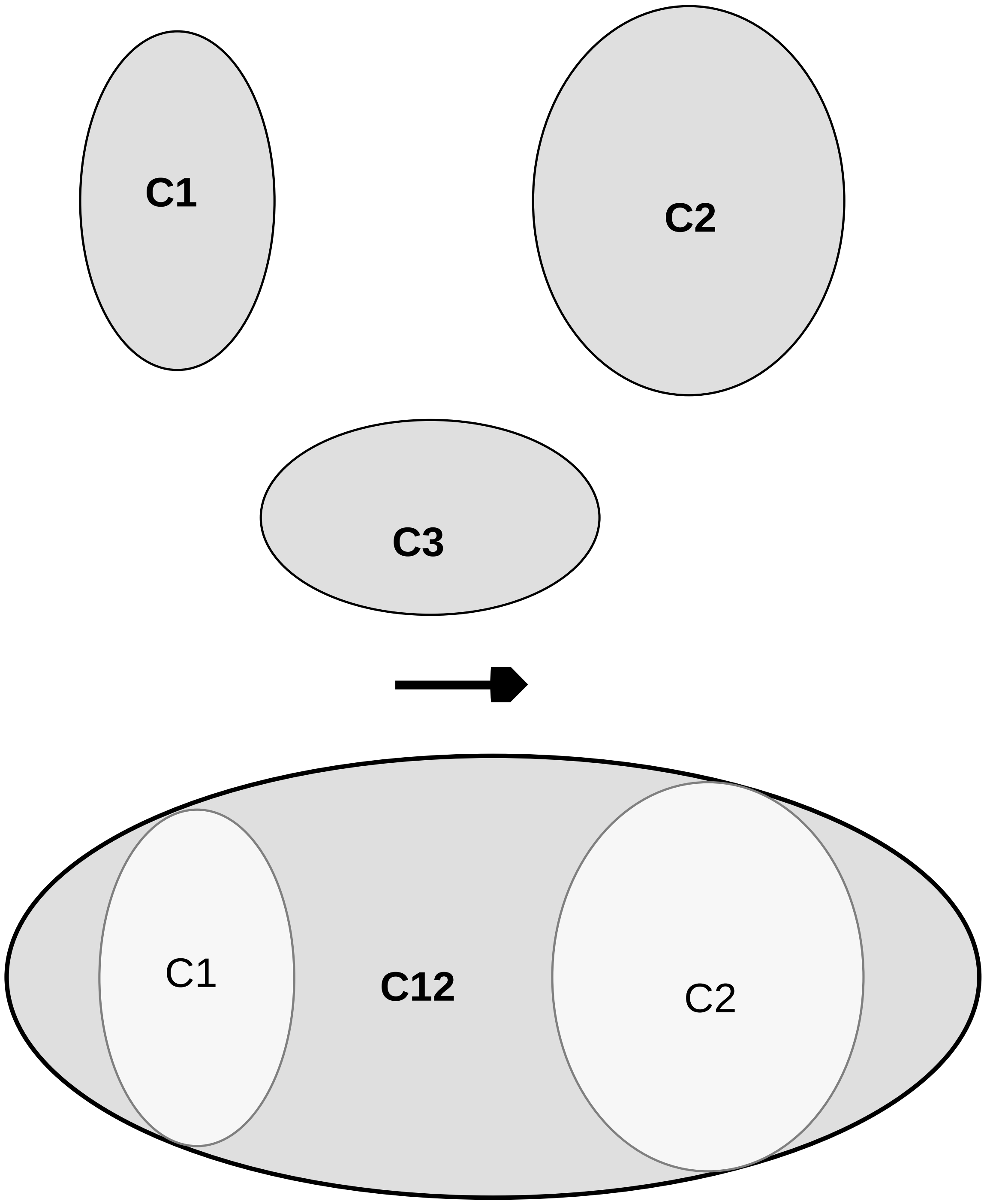
But ...

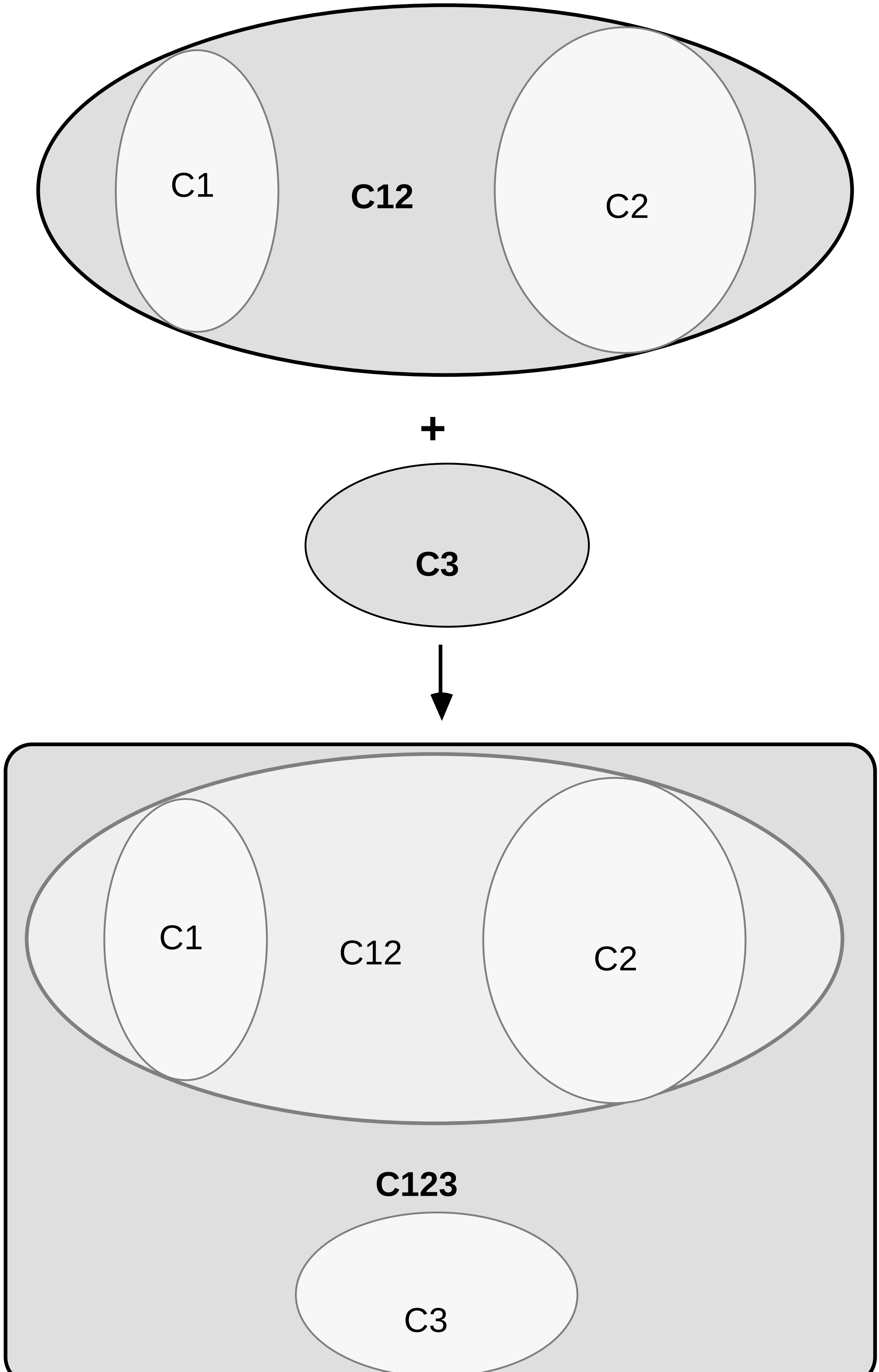
Freely applying the qualitative resolution rule leads to combinatorial explosion.

Pressure regulator (5 equations) --> hundreds of different ways for the resolution rule to apply

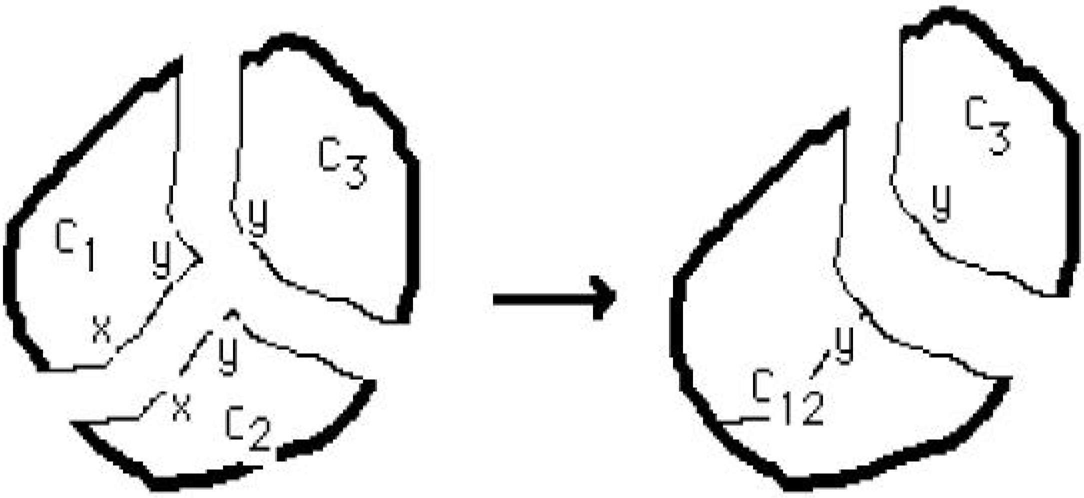
How to control qualitative resolution ?

Consolidation
[Bylander, 1987]




C1
C12
C2
+
C3
C1
C12
C2
C123
C3

Joining two components



y must appear in a model of C_{12}, but x should not.

Composing behavioral descriptions of C_1 and C_2 into an equivalent model for C_{12} requires to eliminate the variables (like x) involved only in C_1 and C_2.

The joining rule

Let E be a set of confluences and x a variable involved in exactly two confluences. If the resolution rule applies to confluences E_1 and E_2 by eliminating variable x, and if x is exclusively involved in E_1 and E_2, then choose this application.
An equivalent model (as far as variables different from x are concerned) is obtained by substituting confluence E_{12} produced in this way for confluences E_1 and E_2.

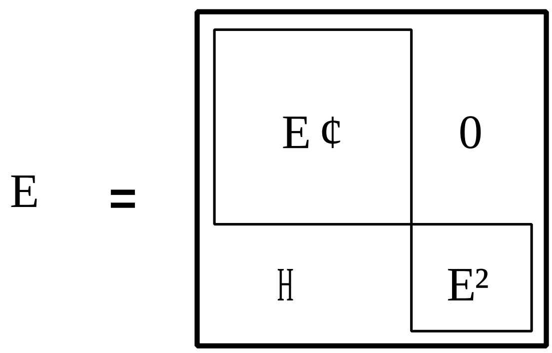
This rule can apply recursively: a variable y involved solely in E_1, E_2 and another confluence belongs to exactly two confluences after the joining rule has been fired.

A mathematical justification

• **The conclusion -** ***substituting E_{12} for E_1 and E_2 provides an equivalent model*** **- has been proved in the square case. Indeed, we proved that any piece of assemblage that can be drawn from the initial model can be drawn after the joining rule has been fired as well.**

• **We proved more** ***(negative part of the joining rule)*****:**

Let $\mathcal{E}$ be a non decomposable set of confluences, and x a variable involved in exactly two confluences E_1 and E_2. If the resolution rule does not apply by eliminating x, then no piece of assemblage involving a variable different from x can be drawn from $\mathcal{E}$.



When can the joining rule fail?

Though working properly in various examples, the joining rule *is not* complete: there are sets of confluences that can be assembled but have *no* variable involved in *less than 3* confluences.

Example:

$$
\begin{array}{ccccccc}
 & & y & + z & + t & \sim & 0 \\
x & & & - z & + t & \sim & 0 \\
x & + y & & & - t & \sim & 0 \\
x & - y & & + z & & \sim & u
\end{array}
$$

can be assembled in

$$
\begin{array}{l}
x \sim u \\
y \sim -u \\
z \sim u \\
t \sim ?u
\end{array}
$$

Signed
maximal
non decomposable
canonical
qualitative matrices

(SMNDQM)

Signed = Stationary (or determinant = + or -)

Maximal = The matrix becomes unsigned as soon as one replaces a 0 entry by a + or - entry.

Two matrices are equivalent iff they can be mapped on each other by composing the operators:

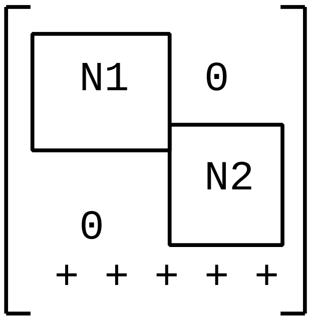
- **exchanging two rows/two columns**
- **multiplying a row/a column by -**
- **transpose**

One selects a *canonical* representative from a class of equivalent matrices.

--> mathematical economists

$$
\begin{bmatrix}
+ & - & 0 & . & . & . & 0 \\
+ & + & - & 0 & . & . & 0 \\
. &   & . & . & . &   & . \\
. &   &   & . & . & . & . \\
+ &   &   & + & + & - & 0 \\
+ & . & . & . & + & + & - \\
+ & . & . & . & . & + & +
\end{bmatrix}
$$

<u>Lancaster's matrices</u>



<u>Gorman's matrices</u>

$$
\begin{bmatrix}
0 & + & + & + \\
+ & 0 & - & + \\
+ & + & 0 & - \\
+ & - & + & 0
\end{bmatrix}
$$

$$\begin{bmatrix} + & - & 0 & 0 & 0 \\ + & + & - & 0 & 0 \\ + & + & + & - & 0 \\ + & + & + & + & - \\ + & + & + & + & + \end{bmatrix} \quad \begin{bmatrix} + & - & 0 & 0 & 0 \\ + & + & - & 0 & 0 \\ + & + & + & - & - \\ 0 & 0 & 0 & + & - \\ + & + & + & + & + \end{bmatrix} \quad \begin{bmatrix} + & - & 0 & 0 & 0 \\ + & + & - & + & 0 \\ + & + & + & - & 0 \\ 0 & 0 & + & + & - \\ 0 & 0 & + & + & + \end{bmatrix}$$

$$\begin{bmatrix} + & - & 0 & 0 & 0 \\ + & + & - & - & 0 \\ + & + & + & - & 0 \\ + & + & 0 & + & - \\ + & + & 0 & + & + \end{bmatrix} \quad \begin{bmatrix} + & - & 0 & 0 & 0 \\ + & + & - & + & 0 \\ + & + & + & 0 & - \\ + & + & 0 & + & - \\ 0 & 0 & + & + & + \end{bmatrix} \quad \begin{bmatrix} + & - & - & 0 & 0 \\ + & + & 0 & + & 0 \\ + & 0 & + & 0 & - \\ - & 0 & 0 & + & - \\ 0 & - & + & + & + \end{bmatrix}$$

The six 5x5 signed maximal non decomposable qualitative matrices

Implementation issues

Basic machinery

Let $\mathcal{E}_0$ be the set of confluences to be assembled.

<u>Choice, step i:</u> Select from $\mathcal{E}_i$ a variable x such that

- **x is involved in exactly two equations of $\mathcal{E}_i$.**
- **x has not been yet selected at step i**
- **there is a variable different from x involved in $\mathcal{E}_i$ which has not been yet assembled.**

<u>Joining rule, step i:</u>
Apply the resolution rule to x, E_1 and E_2. Set $\mathcal{E}_{i+1}$ <-- $\mathcal{E}_i$ - $\{E_1,E_2\}$ » $\{E_{12}\}$

<u>Backtracking, step i:</u> Make a new choice, step i, or go back to step i-1.

Simplification rules (I)

<u>Equality rule</u>:

Let $ax+by\,\tilde{}\,0$ (e) be a confluence. Then x=-aby, and -aby can be substituted for x in all the confluences x belongs to.

Example:

$$?P_4+?A\,\tilde{}\,0 \quad \text{-->} \quad ?A=-?P_4$$

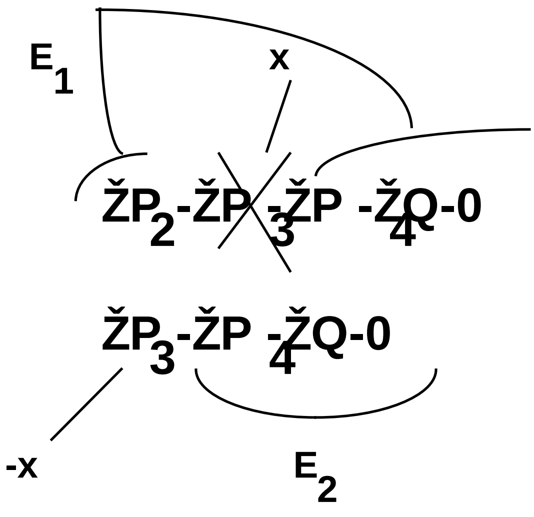
and

$$?P_2-?P_3-?Q+?A\,\tilde{}\,0 \quad \text{-->}$$
$$?P_2-?P_3-?Q-dP_4\,\tilde{}\,0$$

Simplification rules (II)

Ritschard's rule:
Let $x+E_1$˜ a (C_1) and $-x+E_2$˜ b (C_2) such that E_1 and E_2 have no variable with opposite coefficients in common. <u>Assume that all the variables involved in E_2 are also involved in E_1</u> and that <u>a+b=b</u>. Then, E_3˜ a+b (C_3) is a valid confluence, and substituting (C_3) for (C_1) provides an equivalent set of confluences.

Example:



Simplification rules (III)

<u>Single-occurrence-elimination rule</u>: **If a variable x occurs in a single confluence (e) involving at least two variables, then discard x and (e) until assembling is completed.**

Example:

After previous application of Ritschard's rule, $?P_3$ occurs only in confluence (3). Hence $?P_3$ and (3) can be discarded.

Soft Components

• Several algorithms have been proposed to solve this problem

    ? De Kleer & Brown, 1984

    ? Travé & Kaskurewicz, 1986

• But the structure of the solution set has not been yet investigated.

Non standard qualitative models: Orders of magnitude

Raiman, 1986

- **Let (I,+,=) be a totally ordered commutative group, and $(e_i)_{i \in I}$ be some distinct objects.**

- **$S^* = \{+e_i, -e_i, ?e_i\}_{i \in I} \cup \{0\}$**

- **$s_1 e_i + s_2 e_j = s_1 e_i$ if $i>j$**
  $s_2 e_j$ if $i<j$
  $(s_1+s_2)e_i$ if $i=j$
  $x + 0 = 0 + x = x$

- **$s_1 e_i \,.\, s_2 e_j = (s_1.s_2)e_{i+j}$**
  $x.0 = 0.x = 0$

- **$s_1 e_i \sim s_2 e_j$ iff $s_1 = ?$ and $i>j$**
  or
  $s_2 = ?$ and $i<j$
  or
  $s_1 \sim s_2$ and $i=j$

The Qualitative Resolution Rule for Orders of Magnitude

Let x, y, z, a, b be in S* such that

x + y ˜ a (1)

-x + z ˜ b (2)

If <u>x has the pattern se̦ and if s is different from</u>

<u>?</u>, then

y + z ˜ a + b (3)

Interval algebras

- **Consider (E,^). One defines ^ on P(E) by**
  A^ B = {a^ b; aÎ A and bÎ B}

- **This enables us to define + and * on the set of the real intervals I. One defines ˜ on I by**
  I ˜ J iff I

- If one considers a subset **J of I, one defines**
  I^ ȷJ = Min{KÎ J; K I^ J}

provided that this exists.

- **(S,+,*,˜) is an interval algebra with**
  + =]0,+8 [- =]-8 ,0[
  ? =]-8 ,+8 [0 = [0,0]

- **But, an interval algebra often has awful properties (the addition may be not associative).**

The Qualitative Resolution Rule for Interval Algebras

• ***Let (J,+J,*J,˜) be an interval algebra, and let x, y, z, a, b be elements of J such that***

x + y ˜ a (1)

-x + z ˜ b (2)

Suppose that J is stable under intersection (i.e. that IIƒ x is minimal with respect to inclusion (that is, there exists no x' belonging to J such that xÉx' and x?x'), then

y + z ˜ a + b (3)

Other models

Dubois & Prade, 1988

• One considers three objects S, M and L, which are intended to represent the intervals]0,sm[,]sm,ml[and]ml,+8 [(but the landmarks sm and ml are unknown).

• F = {Set of intervals generated by unioning and multiplying by - the intervals S, L and M} U {0}.

• One can define + in different ways, for instance

S + S = +

or

S + S = S U M

We choose the second definition if we know that 2sm < ml.

• In either case, there is a resolution rule. The condition on x is that it belongs to the set {0,S,M,L,-S,-M,-L} (i.e., is minimal with respect to inclusion).

Why *Resolution* ?

Similar aspect:

Let X, Y, Z be propositional variables (and x, y, z their boolean equivalents) such that

X v Y (x + y = 1)

¬X v Z (-x + y = 1)

Then

Y v Z (y + z = 1)

Completeness properties